MATH 256 Section 202
W2
Midterm Exam 1
February 1, 2017
Time Limit: 45 Minutes

Last Name: instructor

First Name: instructor

Student #: instructor

This exam contains 6 pages (including this cover page) and 4 problems. Enter all requested information on the top of this page.

The following rules apply:

- You may **not** use your books, notes, or any calculator on this exam.
- Unless a question asks you to state, or write down the answer, you must **show all your working**. There is credit given for using the correct method to solve each problem. **Unsupported final answers will not receive full credit.**
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering is very difficult to mark and might receive reduced credit if your argument is not clear.
- If you need more space, use the back of the pages.

Do not write in the table to the right.

Problem	Points	Score
1	4	
2	9	
3	6	
4	13	
Total:	32	

Do not open the exam until instructed to do so.

1. For each of the following differential equations for the function $y(x)$, state the order of the differential equation and state whether it is linear or nonlinear.

(a) (2 points)

$$\frac{dy}{dx} + xy = x^2$$

linear, order 1
[1] [1]

(b) (2 points)

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + x^3y = e^y$$

nonlinear, order 2
[1] [1]

2. (a) (3 points) What is the integrating factor for the following differential equation for $y(x)$?

$$\tan(x)\frac{dy}{dx} + y = f(x)$$

Hint:

$$\int \cot(x)\,dx = \ln(\sin(x)) + C \text{ where C is an arbitrary constant.}$$

$$\tan(x)\frac{dy}{dx} + y = f(x)$$

$$\Rightarrow \frac{dy}{dx} + \cot x\, y = f(x)$$

So $p(x) = \cot x$ [1]

and $\int p\,dx = \ln(\sin x)$
(using hint) [1]

So integrating factor is:

$$q(x) = \exp\left(\int p\,dx\right) = \exp(\ln(\sin x))$$
$$= \sin x.$$ [1]

(b) (6 points) What is the general solution to the following differential equation?

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 2x$$

We have $p(x) = 2x$

So $\int p\,dx = x^2$

The integrating factor is

$q(x) = \exp\left(\int p\,dx\right) = e^{x^2}$ [2]

So $\frac{d}{dx}\left(e^{x^2} y\right) = 2x e^{x^2}$ [2]

$\Rightarrow e^{x^2} y = \int 2x e^{x^2}\,dx = e^{x^2} + C$

So $y = 1 + Ce^{-x^2}$ [2]

with C = const.

3. (6 points) Solve the following differential equation for $y(x)$ with boundary condition $y(0) = 1$. Leave your answer as an implicit equation involving y and x.

$$\frac{dy}{dx} = \frac{e^{3x}}{y-2}$$

The equation is seperable:

$$(y-2)\frac{dy}{dx} = e^{3x} \qquad [2]$$

$$\Rightarrow \int (y-2)\,dy = \int e^{3x}\,dx$$

$$\Rightarrow \frac{1}{2}y^2 - 2y = \frac{1}{3}e^{3x} + C \qquad [2]$$

Using $y(0)=1$, we have

$$\frac{1}{2} - 2 = \frac{1}{3} + C$$

So $C = \frac{-11}{6}$

and $\frac{1}{2}y^2 - 2y = \frac{1}{3}e^{3x} - \frac{11}{6}$. [2]

4. Consider the following differential equation for $y(x)$:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = f(x).$$

(a) (3 points) What is the homogeneous solution $y_h(x)$?

Try $y = e^{\lambda x}$. Then $\lambda^2 + 2\lambda + 1 = 0$ [1]

$\Rightarrow (\lambda+1)^2 = 0 \Rightarrow \lambda = -1$ (twice) [1]

So $y_h(x) = (Ax + B)e^{-x}$. [1]

(b) (5 points) For each of the following functions $f(x)$, *write down* the form of the particular solution $y_p(x)$ that will solve the differential equation considered in part (a). You do not need to solve for the unknown coefficients.

i. $f(x) = 2x + 3$

$y_p(x) = Cx + D$ [1]

ii. $f(x) = 5\sin(2x)$

$y_p(x) = C\sin 2x + D\cos 2x$ [1]

iii. $f(x) = 6e^{2x}$

$y_p(x) = Ce^{2x}$ [1]

iv. $f(x) = (3x + 2)e^{x}$

$y_p(x) = (Cx + D)e^{x}$ [1]

v. $f(x) = 4e^{-x}$

$y_p(x) = (Cx^2 + Dx)e^{-x}$. [1]

(c) (5 points) What is the general solution to the following differential equation for $y(x)$?

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2x + 3$$

Hint: This is the same equation considered in parts (a) and (b) with the function f(x) given in (b) part (i).

The solution is

$$y(x) = y_h(x) + y_p(x)$$
$$= (Ax+B)e^{-x} + y_p(x) \text{ using (a). } [1]$$

Then we try $y_p(x) = Cx + D$. [1]

So $y_p' = C$ and $y_p'' = 0$.

Then, $y_p'' + 2y_p' + y_p$

$= 0 + 2C + Cx + D$

$= 2x + 3.$

Coeff x^1: $C = 2$.

Coeff x^0: $2C + D = 3 \Rightarrow D = 3 - 2C = -1$ [2]

Hence $y(x) = (Ax+B)e^{-x} + 2x - 1$ [1]