MATH 256 Section 202
W2
Midterm Exam 1
February 9, 2018
Time Limit: 45 Minutes

Last Name: instructor

First Name:

Student #:

This exam contains 8 pages (including this cover page) and 6 problems. Enter all requested information on the top of this page.

The following rules apply:

- You may **not** use your books, notes, or any calculator on this exam.
- Unless a question asks you to state, or write down the answer, you must **show all your working**. There is credit given for using the correct method to solve each problem. **Unsupported final answers will not receive full credit.**
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering is very difficult to mark and might receive reduced credit if your argument is not clear.
- If you need more space, use the back of the pages.

Do not write in the table to the right.

Problem	Points	Score
1	6	
2	6	
3	3	
4	6	
5	7	
6	10	
Total:	38	

Do not open the exam until instructed to do so.

1. (6 points) For each of the following differential equations for the function $y(x)$, state the order of the differential equation and state whether it is linear or nonlinear.

   (a) (2 points)

$$\frac{d^7y}{dx^7} + y = \tan^2(x)$$

order 7 linear (2)

   (b) (2 points)

$$\frac{d^3y}{dx^3} + \frac{dy}{dx} + xy^2 = x^6$$

order 3 nonlinear (2)

   (c) (2 points)

$$\frac{dy}{dx} + \ln(x)y = \ln(y)$$

order 1 nonlinear (2)

2. (6 points) Solve the following differential equation for $y(x)$ with the condition $y\left(\frac{\pi}{2}\right) = 1$.

$$x\frac{dy}{dx} + 2y = \sin(x)$$

Hint:

$$\int x\sin(x)\,dx = \sin(x) - x\cos(x) + C \text{ where C is an arbitrary constant.}$$

State the interval of existence of the solution.

$$y' + \frac{2}{x}y = \frac{\sin x}{x}.$$

Int. fac. is $q = \exp\left(\int \frac{2}{x}dx\right) = \exp(2\ln x) = x^2$ (1)

$\Rightarrow (x^2y)' = x\sin x$ (1)

$\Rightarrow x^2y = \sin x - x\cos x + C$ using hint (1)

$\Rightarrow y = \frac{\sin x}{x^2} - \frac{\cos x}{x} + \frac{C}{x^2}.$ (1)

$y(\frac{\pi}{2}) = 1 \Rightarrow 1 = \frac{4}{\pi^2} + \frac{4C}{\pi^2} \Rightarrow C = \frac{\pi^2}{4} - 1.$ (1)

Interval $0 < x < \infty$ (1)

(more space for question 2):

3. (3 points) Solve the following differential equation for $y(x)$ with $y(0) = 1$. Leave your answer as an implicit equation involving y and x.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - 1}{3 + 2y}$$

Seperable: $(3+2y)\,y' = 3x^2 - 1$ 1

$\Rightarrow \quad 3y + y^2 = x^3 - x + C$ 1

$y(0) = 1 \Rightarrow C = 4.$ 1

4. (6 points) Consider the following differential equation for $y(x)$.

$$\frac{d^2y}{dx^2} + y = f(x).$$

(a) (3 points) What is the homogeneous solution $y_h(x)$?

Try $y_h = e^{\lambda x} \Rightarrow \lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm i$ 2

$\Rightarrow y_h = A\cos x + B\sin x.$ 1.

(b) (4 points) For each of the following functions $f(x)$, *write down* the form of the particular solution $y_p(x)$ that will solve the differential equation considered in part (a). You do not need to solve for the unknown coefficients.

i. (1 point) $f(x) = 2x + 7$

$Cx + D$ 1

ii. (1 point) $f(x) = \sin(3x)$

$C\cos 3x + D\sin 3x$ 1

iii. (1 point) $f(x) = 5e^x$

Ce^x 1

iv. (1 point) $f(x) = \cos(x)$

$x(C\cos x + D\sin x)$ 1

5. (7 points) Consider the differential equation

$$x^2\frac{d^2y}{dx^2} - x(x+2)\frac{dy}{dx} + (x+2)y = 0.$$

(a) (3 points) Use Abel's formula to find the Wronskian up to a multiplicative constant.

$$y'' - \left(1+\frac{2}{x}\right)y' + \left(\frac{1}{x}+\frac{2}{x^2}\right)y = 0 \quad (1)$$

$$W = A\exp\left(-\int -\left(1+\frac{2}{x}\right)dx\right) = A\exp(x+2\ln x) = Ax^2e^x$$

(1) (1)

(b) (4 points) Verify that $y_1(x) = x$ and $y_2(x) = xe^x$ are a fundamental set of solutions for the equation. if $x>0$.

$y_1' = 1 \quad y_1'' = 0 \quad , x^2y_1'' - x(x+2)y_1' + (x+2)y_1$

$= -x^2 - 2x + x^2 + 2x = 0$ (1)

$y_2' = xe^x + e^x, \; y_2'' = 2e^x + xe^x$

$x^2y_2'' - x(x+2)y_2' + (x+2)y_2 = 2x^2e^x + x^3e^x - x^3e^x - x^2e^x$

$-2x^2e^x - 2xe^x + x^2e^x + 2e^x = 0$ (2)

$W = y_1y_2' - y_1'y_2 = x(xe^x + e^x) - xe^x = x^2e^x \neq 0$ (1)

if $x>0$.

(more space for question 5)

(question 6 follows on next page)

6. (10 points) Find the general solution to the following differential equation for $y(x)$,

$$\frac{d^2y}{dx^2} + 9y = \frac{9}{\cos(3x)}.$$

Hint:

$$\int \tan(3x)\,dx = \frac{1}{3}\ln(\cos(3x)) + C \text{ where C is an arbitrary constant.}$$

$y = y_h + y_p$. (1) Try $y_h = e^{\lambda x} \Rightarrow \lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$ (1)

So $y_h = A\cos 3x + B\sin 3x$. (1)

Let $y_p = u(x)\cos 3x + v(x)\sin 3x$ (1)

with $u'\cos 3x + v'\sin 3x = 0$ (1) (1)

and $-3u'\sin 3x + 3v'\cos 3x = \frac{9}{\cos 3x}$ (2). (1)

(1) $\Rightarrow$ $u' = -v'\tan 3x$

(2) $\Rightarrow$ $3v'\sin 3x\tan 3x + 3v'\cos 3x = \frac{9}{\cos 3x}$ (1)

$\Rightarrow v' = \frac{3}{\cos 3x(\sin^3 x\tan 3x + \cos 3x)} = \frac{3}{\cos^2 3x + \sin^2 3x} = 3$ (1)

$\Rightarrow v = 3x$.

also $u' = -3\tan 3x \Rightarrow u = -\ln(\cos(3x))$ (HINT) (1)

So $y(x) = A\cos 3x + B\sin 3x + 3x\sin 3x - \cos(3x)\ln(\cos 3x)$.

(1)

(more space for question 6)